# Remarks on the confluent KZ equation for $\mathfrak{sl}_2$ and quantum Painlevé equations



2008 J. Phys. A: Math. Theor. 41 175205

(http://iopscience.iop.org/1751-8121/41/17/175205)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 175205 (14pp) doi:10.1088/1751-8113/41/17/175205

# Remarks on the confluent KZ equation for $\mathfrak{sl}_2$ and quantum Painlevé equations

**M Jimbo**[1], **H Nagoya**[2] **and J Sun**[1]

[1] Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo 153-8914, Japan
[2] Department of Mathematics, School of Fundamental Science and Technology, Keio University, 3-14-1, Hiyoshi, MinatoKita, Yokohama, Kanagawa, 223-8522 Japan

E-mail: jimbomic@ms.u-tokyo.ac.jp, nagoya@math.keio.ac.jp and sunjuan@ms.u-tokyo.ac.jp

Received 3 February 2008
Published 15 April 2008
Online at stacks.iop.org/JPhysA/41/175205

**Abstract**
For the Lie algebra $\mathfrak{sl}_2$, we give an explicit form of an irregular-singular version of the KZ equation. In the simplest non-trivial cases these KZ equations reproduce the quantum Painlevé equations $\mathrm{QP_{II}}$–$\mathrm{QP_V}$ possessing affine Weyl group symmetry, found previously by one of the authors.

PACS numbers: 02.30.Hq, 02.20.Sv, 02.30.Ik

## 1. Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra, and let $V_1, \ldots, V_n$ be $\mathfrak{g}$-modules. The Knizhnik–Zamolodchikov (KZ) equation is a system of linear differential equations for the unknown function $u \in V_1 \otimes \cdots \otimes V_n$,

$$\kappa \frac{\partial u}{\partial z_i} = \sum_{j(\neq i)} \frac{\Omega^{(i,j)}}{z_i - z_j} u \qquad (i = 1, \ldots, n), \tag{1.1}$$

where $\kappa$ is a parameter and $\Omega^{(i,j)}$ stands for the quadratic Casimir element of $U\mathfrak{g}$ acting non-trivially on the $i$th and $j$th tensor factors. Operators appearing on the right-hand side of (1.1) are the Gaudin Hamiltonians well known and studied in integrable systems.

It was observed some time ago [1, 2] that the KZ equation gives a quantum version of the monodromy preserving deformation of linear differential equations. The interpretation goes as follows. Let $U = U(z_1, \ldots, z_n)$ be a fundamental solution of (1.1). Consider also the KZ equation which has one more singularity at $z_0 = z$, and let $\tilde{U}(z) = \tilde{U}(z, z_1, \ldots, z_n)$ be a fundamental solution with values in $\mathrm{End}(V_0) \otimes \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$. Then $Y(z) = U^{-1}\tilde{U}(z)$ satisfies

$$\frac{\partial Y}{\partial z} = \sum_{i=1}^{n} \frac{A_i}{z - z_i} Y, \tag{1.2}$$

$$\frac{\partial Y}{\partial z_i} = -\frac{A_i}{z - z_i} Y, \tag{1.3}$$

where $A_i = (1/\kappa)U^{-1}\Omega^{(0,i)}U$. Equations (1.2), (1.3) are identical in form to the auxiliary linear system of the Schlesinger equations. Hence the same equations hold in the present case for the matrices $A_i$ with non-commutative entries. The classical case is recovered in the limit $\hbar = 1/\kappa \to 0$.

Monodromy preserving deformation of linear differential equations with irregular singularities was studied in [3]. In addition to the position of the poles $z_i$, there appear new deformation parameters (the parameters of irregularity). In [1] the case with multiple poles was also treated, but equations with respect to the parameters of irregularity have not been discussed. The case with poles of order at the most two was studied later from different points of view in [4–6]. Recently, Gaudin Hamiltonians with irregular singularities have attracted attention in connection with the geometric Langlands correspondence [7, 8].

To the authors' knowledge, the KZ equation with respect to the parameters of irregularity has not been written explicitly in the literature beyond the case of Poincaré rank 1 (i.e., double poles). The aim of the present note is to do that in the simplest case $\mathfrak{g} = \mathfrak{sl}_2$. From the argument of [1], it is well expected that the corresponding isomonodromy deformation equations lead to a quantization of Painlevé and degenerate Garnier equations. Indeed, in the simplest cases we find the quantum Painlevé equations with affine Weyl group symmetry of types II through V (to be denoted by $\mathrm{QP_{II}}$–$\mathrm{QP_V}$), which have been introduced previously[3] by one of the authors [9].

The text is organized as follows. In section 2, we summarize some facts about the Gaudin Hamiltonians associated with a Lax operator having poles of arbitrary order. In section 3, we introduce confluent Verma modules for $\mathfrak{sl}_2$. In section 4, we define the confluent KZ equation. We also give integral formulae for solutions. In section 5, we interpret them as quantization of monodromy preserving deformation from KZ equation following the recipe of [1]. In particular, we identify the Hamiltonians of the quantum (irregular) Schlesinger-type systems with the (generalized) Gaudin Hamiltonians. In section 6, we discuss examples of the quantum Painlevé equations $\mathrm{QP_I}$–$\mathrm{QP_V}$.

We note here that the quantum Schlesinger-type system and the quantum Painlevé equations are not quite equivalent. Certainly the latter is obtained from the former by eliminating some entries of the associated linear equation, but in this process it is necessary to invert them. Namely we are forced to work in the quotient skew field generated by these non-commutative elements. For the KZ equation associated with highest weight-type modules, this invertibility fails. Hence our integral solutions to the KZ equation do not give rise to solutions to the quantum Painlevé equations. It remains an open question to find solutions to the latter.

Throughout this note, we set $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$, and denote by $e, f, h$ the standard basis of $\mathfrak{g}$. For a non-negative integer $r$, we denote by $\mathfrak{g}_{(r)} = \mathfrak{g}[t]/t^{r+1}\mathfrak{g}[t]$ the Lie algebra of truncated polynomials over $\mathfrak{g}$.

## 2. Gaudin Hamiltonians

In this section, we collect necessary facts about the Gaudin Hamiltonians for $\mathfrak{sl}_2$ associated with a rational $L$ operator. The materials in this section are fairly standard. For the connection with conformal coinvariants (see, e.g., [7]).

[3] The case $\mathrm{QP_{III}}$ has not been considered in [9]. For comparison we also consider $\mathrm{QP_I}$ which has no affine Weyl group symmetry.

Let $z_1, \dots, z_n, \infty$ be distinct points on $\mathbb{P}^1$, and let $r_1, \dots, r_n, r_\infty$ be non-negative integers. Set $\mathfrak{a} = \left(\oplus_{i=1}^n \mathfrak{g}^{(i)}\right) \oplus \mathfrak{g}^{(\infty)}$, where

$$\mathfrak{g}^{(i)} = \mathfrak{g}[t]/t^{r_i+1}\mathfrak{g}[t] \quad (1 \leqslant i \leqslant n), \qquad \mathfrak{g}^{(\infty)} = t\mathfrak{g}[t]/t^{r_\infty+1}\mathfrak{g}[t].$$

Let further $\tilde{\mathfrak{a}} = \mathfrak{g}^{(0)} \oplus \mathfrak{a}$, where $\mathfrak{g}^{(0)} = \mathrm{End}(\mathbb{C}^2)$ and $\mathbb{C}^2$ is the standard two-dimensional representation of $\mathfrak{g}$. Denoting by $x_p^{(i)}$ the image of $x \otimes t^p$ under the embedding $\mathfrak{g}^{(i)} \to \tilde{\mathfrak{a}}$, we set

$$\Omega_{p,q}^{(i,j)} = e_p^{(i)} f_q^{(j)} + f_p^{(i)} e_q^{(j)} + \tfrac{1}{2} h_p^{(i)} h_q^{(j)}.$$

The following element of $\tilde{\mathfrak{a}}$ is called an $L$ operator:

$$L(z) = \sum_{i=1}^n \sum_{p=0}^{r_i} \frac{\Omega_{0,p}^{(0,i)}}{(z - z_i)^{p+1}} - \sum_{p=1}^{r_\infty} z^{p-1} \Omega_{0,p}^{(0,\infty)}.$$

It satisfies the commutation relation

$$[L(z) \otimes 1, 1 \otimes L(w)] = -\left[\frac{P}{z - w}, L(z) \otimes 1 + 1 \otimes L(w)\right], \tag{2.1}$$

where $P \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ stands for the transposition. Define

$$\Delta(z) = \tfrac{1}{2}\mathrm{tr}(L(z)^2).$$

The commutation relation (2.1) implies

$$[L(z), \Delta(w)] = \frac{[L(z), L(w)]}{z - w}, \tag{2.2}$$

$$[\Delta(z), \Delta(w)] = 0. \tag{2.3}$$

Introducing the Laurent coefficients $G_k^{(\lambda)} \in U(\mathfrak{a})$ by

$$\Delta(z) = \sum_{k=-\infty}^{2r_i} G_k^{(i)} (z - z_i)^{-k-2} \qquad (z \to z_i),$$

$$= \sum_{k=-\infty}^{2r_\infty} G_k^{(\infty)} z^{k-2} \qquad (z \to \infty),$$

we have

$$\Delta(z) = \sum_{i=1}^n \sum_{p=-1}^{2r_i} \frac{G_p^{(i)}}{(z - z_i)^{p+2}} + \sum_{p=2}^{2r_\infty} G_p^{(\infty)} z^{p-2}.$$

From (2.2) we deduce that

$$G_k^{(i)} (r_i \leqslant k \leqslant 2r_i, i = 1, \dots, n), \qquad G_k^{(\infty)} (r_\infty + 1 \leqslant k \leqslant 2r_\infty) \tag{2.4}$$

are central elements of $U(\mathfrak{a})$, while for the element $G_{r_\infty}^{(\infty)}$ we have

$$\left[G_{r_\infty}^{(\infty)}, L(z)\right] = \left[\Omega_{0,r_\infty}^{(0,\infty)}, L(z)\right]. \tag{2.5}$$

From (2.3), the remaining coefficients

$$\left\{G_k^{(i)} \,\middle|\, -1 \leqslant k \leqslant r_i - 1, i = 1, \dots, n\right\} \cup \left\{G_k^{(\infty)} \,\middle|\, 2 \leqslant k \leqslant r_\infty - 1\right\} \tag{2.6}$$

are mutually commutative. We call them (generalized) Gaudin Hamiltonians. Later these elements will be used to construct the confluent KZ connection. We shall also use

$$G_1^{(\infty)} = \sum_{i=1}^n G_{-1}^{(i)}, \qquad G_0^{(\infty)} = \sum_{i=1}^n \left(z_i G_{-1}^{(i)} + G_0^{(i)}\right), \tag{2.7}$$

which belong to the above commutative family. Explicitly the elements (2.6) are given by

$$G_k^{(i)} = \frac{1}{2}\sum_{\substack{p+q=k\\ p,q\geqslant 0}} \Omega_{p,q}^{(i,i)} - \sum_{p,q\geqslant 0}\binom{p+q}{p} z_i^q \Omega_{k+p+1,p+q+1}^{(i,\infty)}$$
$$+\sum_{j(\neq i)}\sum_{p,q\geqslant 0}\binom{p+q}{p}\frac{(-1)^p}{(z_i-z_j)^{p+q+1}}\Omega_{k+p+1,q}^{(i,j)} \qquad (k\geqslant -1), \tag{2.8}$$

$$G_k^{(\infty)} = \frac{1}{2}\sum_{\substack{p+q=k\\ p,q>0}} \Omega_{p,q}^{(\infty,\infty)} - \sum_{i=1}^{n}\sum_{p,q\geqslant 0}\binom{p+q}{p} z_i^q \Omega_{p,p+q+k}^{(i,\infty)} \qquad (k\geqslant 1). \tag{2.9}$$

Note that for $G_0^{(\infty)}$ we must use (2.7) since the expression (2.9) does not apply.

## 3. Confluent Verma modules

In this section, we introduce confluent Verma modules which will be used in subsequent sections.

Let $\mathfrak{b} = \mathbb{C}e \oplus \mathbb{C}h$, $\mathfrak{b}_{(r)} = \mathfrak{b}[t]/t^{r+1}\mathfrak{b}[t]$. For an $(r+1)$-tuple of parameters $\gamma = (\gamma_0,\ldots,\gamma_{r-1},\gamma_r) \in \mathbb{C}^r\times\mathbb{C}^\times$, consider the induced module

$$M(\gamma) = \mathrm{Ind}_{\mathfrak{b}_{(r)}}^{\mathfrak{g}_{(r)}}\mathbb{C}\mathbf{1}_\gamma, \tag{3.1}$$

where $\mathbb{C}\mathbf{1}_\gamma$ denotes the one-dimensional $\mathfrak{b}_{(r)}$-module given by

$$(e\otimes t^p)\mathbf{1}_\gamma = 0, \qquad (h\otimes t^p)\mathbf{1}_\gamma = \gamma_p\mathbf{1}_\gamma \qquad (0\leqslant p\leqslant r). \tag{3.2}$$

We call (3.1) confluent Verma module of Poincaré rank $r$ with parameter $\gamma$.

We shall consider also a variant of (3.1) for the Lie subalgebra $\mathfrak{g}'_{(r)} = t\mathfrak{g}[t]/t^{r+1}\mathfrak{g}[t]$. Taking $\gamma = (0,\gamma')$ with $\gamma'\in\mathbb{C}^{r-1}\times\mathbb{C}^\times$ and regarding $M(\gamma)$ as $\mathfrak{g}'_{(r)}$-module, we define its subquotient

$$M'(\gamma') = \widetilde{M}'(\gamma')/(f\otimes t^r\cdot\widetilde{M}'(\gamma')), \qquad \widetilde{M}'(\gamma') = U(\mathfrak{g}'_{(r)})\mathbf{1}_\gamma.$$

Since $[h_0,\mathfrak{g}'_{(r)}]\subset\mathfrak{g}'_{(r)}$ the action of $h_0 = h\otimes t^0$ is well defined on $M'(\gamma')$, so we shall regard $M'(\gamma')$ as a module over $\mathfrak{g}'_{(r)}\oplus\mathbb{C}h_0$.

In fact, the module $M(\gamma)$ is obtained from the special case $M(\gamma^0)$ $(\gamma^0 = (\gamma_0,0,\ldots,0,1))$ as a twist by an automorphism,

$$\rho(\gamma_0,\gamma') = \rho(\gamma^0)\circ\varphi(\gamma'),$$

where $\rho(\gamma) : U(\mathfrak{g}_{(r)})\to\mathrm{End}(M(\gamma))$ stands for the structure map. The automorphism $\varphi(\gamma')$ of $\mathfrak{g}_{(r)}$ is defined from the derivations $\{\mathfrak{d}_k\}_{0\leqslant k\leqslant r-1}$ given by $\mathfrak{d}_k(\xi\otimes t^p) = p\xi\otimes t^{p+k}$ $(\xi\in\mathfrak{g})$. Consider the system of linear differential equations for $\mathfrak{g}_{(r)}$-valued functions $\xi_p$,

$$D_k(\xi_p) = p\xi_{k+p} \qquad (0\leqslant k\leqslant r-1), \tag{3.3}$$

where

$$D_k = \sum_{p=1}^{r-k} p\gamma_{k+p}\frac{\partial}{\partial\gamma_p}.$$

These equations are integrable because

$$[\mathfrak{d}_k,\mathfrak{d}_l] = (l-k)\mathfrak{d}_{k+l}, \qquad [D_k,D_l] = (l-k)D_{k+l}$$

hold, where we set $\gamma_p = 0$ for $p > r$ and $\mathfrak{d}_p = 0$, $D_p = 0$ $(p \geqslant r)$. Let $\xi_p(\gamma')$ be the unique solution of (3.3) satisfying the initial condition $\xi_p(0, \ldots, 0, 1) = \xi \otimes t^p$. Then the automorphism $\varphi(\gamma')$ is given by

$$\varphi(\gamma') : \mathfrak{g}_{(r)} \to \mathfrak{g}_{(r)}, \qquad \xi \otimes t^p \mapsto \xi_p(\gamma'). \tag{3.4}$$

**Remark.** The module $M(\gamma^0)$ has a realization on the vector space $\mathbb{C}[x_0, x_1, \ldots, x_r]$ with $\mathbf{1}_\gamma = 1$. The generators act by the following formulae:

$$f \otimes t^k = x_k, \tag{3.5}$$

$$h \otimes t^k = \delta_{k,r} - 2\sum_{p\geqslant 0} x_{p+k}\partial_p + \gamma_0\delta_{k,0}, \tag{3.6}$$

$$e \otimes t^k = \partial_{r-k} - \sum_{p,q\geqslant 0} x_{p+q+k}\partial_p\partial_q + \gamma_0\delta_{k,0}\partial_0. \tag{3.7}$$

Here $\partial_p = \partial/\partial x_p$, and $x_p = \partial_p = 0$ for $p > r$. Similarly we have a realization of $M'(\gamma')$ on $\mathbb{C}[x_1, \ldots, x_{r-1}]$. For example, when $r = 3$ and $\gamma' = (\gamma_1, \gamma_2, \gamma_3)$, solving equations similar to (3.3) we obtain for $M'(\gamma') = \mathbb{C}[x_1, x_2]$,

$$\begin{aligned}
&e_1(\gamma') = \gamma_3^{1/3}\partial_2 + \tfrac{1}{2}\gamma_3^{-1/3}\gamma_2\partial_1, && e_2(\gamma') = \gamma_3^{2/3}\partial_1, && e_3(\gamma') = 0,\\
&f_1(\gamma') = \gamma_3^{1/3}x_1 + \tfrac{1}{2}\gamma_3^{-1/3}\gamma_2 x_2, && f_2(\gamma') = \gamma_3^{2/3}x_2, && f_3(\gamma') = 0,\\
&h_1(\gamma') = -2\gamma_3^{1/3}x_2\partial_1 + \gamma_1, && h_2(\gamma') = \gamma_2, && h_3(\gamma') = \gamma_3.
\end{aligned}$$

In the sequel, these explicit formulae are not relevant. We shall use only equations (3.3).

## 4. Confluent KZ equation

From now on, we regard $M(\gamma)$ as realized on a fixed underlying vector space $M(\gamma_0, 0, \ldots, 1)$, where the action of the generators $\xi \otimes t^p$ obey equation (3.3). Where necessary we exhibit the $\gamma$-dependence explicitly as $\xi_p(\gamma)$.

Retaining the notation of section 2, let us consider a family of $\mathfrak{a}$-modules

$$\mathbf{M}(\boldsymbol{\gamma}) = M^{(1)} \otimes \cdots \otimes M^{(n)} \otimes M^{(\infty)} \tag{4.1}$$

parametrized by $\boldsymbol{\gamma} = (\gamma^{(0)}, \ldots, \gamma^{(\infty)})$, where

$$\begin{aligned}
&M^{(i)} = M(\gamma^{(i)}), && \gamma^{(i)} = \left(\gamma_0^{(i)}, \ldots, \gamma_{r_i}^{(i)}\right) \in \mathbb{C}^{r_i} \times \mathbb{C}^\times,\\
&M^{(\infty)} = M'(\gamma^{(\infty)}), && \gamma^{(\infty)} = \left(\gamma_1^{(\infty)}, \ldots, \gamma_{r_\infty}^{(\infty)}\right) \in \mathbb{C}^{r_\infty - 1} \times \mathbb{C}^\times.
\end{aligned}$$

We set

$$h_0 = \sum_{i=1}^n h_0^{(i)} + h_0^{(\infty)}.$$

On (4.1), the central elements (2.4) act as the scalar

$$G_k^{(\lambda)} = \beta_k^{(\lambda)} \cdot \mathrm{id} \qquad (r_\lambda \leqslant k \leqslant 2r_\lambda, (\lambda, k) \neq (\infty, r_\infty)), \tag{4.2}$$

$$\beta_k^{(\lambda)} = \frac{1}{4}\sum_{\substack{p+q=k\\ p,q\geqslant\delta_{\lambda,\infty}}} \gamma_p^{(\lambda)}\gamma_q^{(\lambda)} + \left(\frac{k+1}{2} - \delta_{\lambda,\infty}\right)\gamma_k^{(\lambda)}. \tag{4.3}$$

In the exceptional case $(\lambda, k) = (\infty, r_\infty)$ we have in place of (4.2)

$$G_{r_\infty}^{(\infty)} + \tfrac{1}{2}\gamma_{r_\infty}^{(\infty)} h_0 = \beta_{r_\infty}^{(\infty)} \cdot \mathrm{id},$$

which follows from

$$\left[G^{(\infty)}_{r_\infty}, L(z)\right] = \left[\tfrac{1}{2}\gamma^{(\infty)}_{r_\infty}\sigma^3, L(z)\right] = -\left[\tfrac{1}{2}\gamma^{(\infty)}_{r_\infty}h_0, L(z)\right].$$

In the last line $\sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and we used

$$[L(z), h_0 + \sigma^3] = 0.$$

Fixing a parameter $\kappa$, we define the differential operators

$$\Gamma^{(i)}_k = \begin{cases} -\kappa \dfrac{\partial}{\partial z_i} + G^{(i)}_{-1} & (k = -1),\\ -\kappa D^{(i)}_k + G^{(i)}_k - \beta^{(i)}_k & (0 \leqslant k \leqslant r_i - 1), \end{cases} \tag{4.4}$$

$$\Gamma^{(\infty)}_k = \begin{cases} -\kappa D^{(\infty)}_0 + G^{(\infty)}_0 - \dfrac{1}{4}h_0(h_0+2) & (k = 0),\\ -\kappa D^{(\infty)}_k + G^{(\infty)}_k - \beta^{(\infty)}_k + \dfrac{1}{2}h_0\cdot\gamma^{(\infty)}_k & (1 \leqslant k \leqslant r_\infty - 1). \end{cases} \tag{4.5}$$

The additional terms $\beta^{(\lambda)}_k$, $(1/2)h_0\cdot\gamma^{(\infty)}_k$ are introduced in order to ensure that all but a finite number of the operators vanish,

$$\Gamma^{(\lambda)}_k = 0 \qquad (k \geqslant r_\lambda).$$

The term $(1/4)h_0(h_0+2)$ in $D^{(\infty)}_0$ is inserted for convenience.

Consider now the system of linear differential equations

$$\Gamma^{(i)}_{-1}u = 0 \qquad (1 \leqslant i \leqslant n), \tag{4.6}$$

$$\Gamma^{(\lambda)}_k u = 0 \qquad (0 \leqslant k \leqslant r_\lambda - 1, \lambda = 1, \ldots, n, \infty). \tag{4.7}$$

Here $u$ is a $\mathbf{M}(\boldsymbol{\gamma})$-valued unknown function in the variables $\mathbf{z} = (z_1, \ldots, z_n)$ and $\boldsymbol{\gamma}$. We call (4.6), (4.7) the confluent KZ equation. Since $\Gamma^{(\lambda)}_k$'s commute with $h_0$, we may consider them on each eigenspace $\mathbf{M}(\boldsymbol{\gamma})_d$ of $h_0$ with eigenvalue $\sum_{i=1}^n \gamma^{(i)}_0 - 2d$. We note also that (4.6), (4.7) imply the homogeneity relation

$$\kappa\left(D^{(\infty)}_0 - \sum_{i=1}^n\left(D^{(i)}_0 + z_i\frac{\partial}{\partial z_i}\right)\right)u = \left(\sum_{i=1}^n \beta^{(i)}_0 - \frac{h_0}{2}\left(\frac{h_0}{2}+1\right)\right)u.$$

The following lemma shows that this system is integrable.

**Proposition 4.1.** *We have*

$$\left[\Gamma^{(i)}_{-1}, \Gamma^{(j)}_{-1}\right] = 0 \qquad (1 \leqslant i, j \leqslant n),$$

$$\left[\Gamma^{(i)}_{-1}, \Gamma^{(\lambda)}_k\right] = 0 \qquad (0 \leqslant k, 1 \leqslant i \leqslant n, \lambda = 1, \ldots, n, \infty),$$

$$\left[\Gamma^{(\lambda)}_k, \Gamma^{(\mu)}_l\right] = \delta_{\lambda,\mu}\kappa(k-l)\Gamma^{(\lambda)}_{k+l} \qquad (0 \leqslant k, l, \lambda, \mu = 1, \ldots, n, \infty).$$

**Proof.** The assertion follows from the commutativity of the Hamiltonians (2.6), along with the following relations which can be verified directly:

$$\left[D^{(\lambda)}_k, G^{(i)}_{-1}\right] = \frac{\partial}{\partial z_i}G^{(\lambda)}_k,$$

$$\left[D^{(\lambda)}_k, D^{(\mu)}_l\right] = \delta_{\lambda,\mu}(l-k)D^{(\lambda)}_{k+l},$$

$$\left[D^{(\lambda)}_k, G^{(\mu)}_l\right] - \left[D^{(\mu)}_l, G^{(\lambda)}_k\right] = \delta_{\lambda,\mu}(l-k)G^{(\lambda)}_{k+l},$$

$$\left[D^{(\lambda)}_k, \beta^{(\mu)}_l\right] - \left[D^{(\mu)}_l, \beta^{(\lambda)}_k\right] = \delta_{\lambda,\mu}(l-k)\beta^{(\lambda)}_{k+l},$$

$$\left[D^{(\lambda)}_k, \gamma^{(\mu)}_l\right] - \left[D^{(\mu)}_l, \gamma^{(\lambda)}_k\right] = \delta_{\lambda,\mu}(l-k)\gamma^{(\lambda)}_{k+l}.$$

□

The known integral formula for solutions for the KZ equation [13, 14] has a straightforward generalization to the present setting. Define the master function

$$\Phi(\mathbf{t};\mathbf{z},\gamma)=\Phi_1(\mathbf{z},\gamma)^{1/(2\kappa)}\Phi_2(\mathbf{t};\mathbf{z},\gamma)^{1/\kappa},$$

where $\mathbf{t}=(t_1,\ldots,t_d)$ and

$$\Phi_1(\mathbf{z},\gamma)=\prod_{1\leqslant i<j\leqslant n}\left\{(z_i-z_j)^{\gamma_0^{(i)}\gamma_0^{(j)}}\exp\left(-\sum_{\substack{p,q\geqslant 0\\ p+q>0}}\binom{p+q}{p}\frac{1}{p+q}\frac{(-1)^p}{(z_i-z_j)^{p+q}}\gamma_p^{(i)}\gamma_q^{(j)}\right)\right\}$$
$$\times\exp\left(-\sum_{i=1}^n\sum_{\substack{p,q\geqslant 0\\ p+q>0}}\binom{p+q}{p}\frac{1}{q+1}z_i^q\gamma_p^{(i)}\gamma_{p+q}^{(\infty)}\right),$$

$$\Phi_2(\mathbf{t};\mathbf{z},\gamma)=\prod_{1\leqslant a<b\leqslant d}(t_a-t_b)^2\prod_{a=1}^d\prod_{i=1}^n\left\{(t_a-z_i)^{-\gamma_0^{(i)}}\exp\left(\sum_{p>0}\frac{1}{p}\frac{\gamma_p^{(i)}}{(t_a-z_i)^p}\right)\right\}$$
$$\times\exp\left(\sum_{a=1}^d\sum_{p>0}\frac{t_a^p}{p}\gamma_p^{(\infty)}\right).$$

Set further for $x\in\mathfrak{g}$

$$x(t;\mathbf{z},\gamma)=\sum_{i=1}^n\sum_{p\geqslant 0}\frac{x_p^{(i)}(\gamma^{(i)})}{(t-z_i)^{p+1}}-\sum_{p\geqslant 1}x_p^{(\infty)}(\gamma^{(\infty)})t^{p-1}.$$

Let $\mathbf{1}_\gamma$ stand for the tensor product of $\mathbf{1}_{\gamma^{(\lambda)}}$'s.

**Proposition 4.2.** *With an appropriate choice of cycles* $\Gamma$*, the function*

$$u=\int_\Gamma\prod_{a=1}^d dt_a\cdot\Phi(\mathbf{t};\mathbf{z},\gamma)\prod_{a=1}^d f(t_a;\mathbf{z},\gamma)\cdot\mathbf{1}_\gamma$$

*taking values in* $\mathbf{M}(\gamma)_d$ *is a solution to the confluent KZ equation.*

**Proof.** First we recall a formula from algebraic Bethe ansatz for the Gaudin system. We abbreviate $x(t;\mathbf{z},\gamma)$ to $x(t)$. Denote by $\lambda(z)$ the eigenvalue of $h(z)$ on $\mathbf{1}_\gamma$, and set $\tilde\lambda(z)=\lambda(z)-\sum_{a=1}^d 2/(z-t_a)$. Noting that

$$[x(z),y(w)]=-\frac{1}{z-w}([x,y](z)-[x,y](w)),$$
$$[\Delta(z),f(t)]=-\frac{1}{t-z}(f(z)h(t)-f(t)h(z)),$$

we obtain

$$(\Delta(z)-\Lambda(z))\prod_{a=1}^d f(t_a)\mathbf{1}_\gamma=\sum_{a=1}^d\left(\lambda(t_a)-\sum_{b(\neq a)}\frac{2}{t_a-t_b}\right)\frac{f(z)}{z-t_a}\prod_{b(\neq a)}^d f(t_b)\mathbf{1}_\gamma,$$

where $\Lambda(z)=\frac{1}{4}\tilde\lambda(z)^2-\frac{1}{2}\tilde\lambda'(z)$. This formula allows us to find the action of the Gaudin Hamiltonians on the integrand. The assertion can then be verified by a direct substitution. □

## 5. Monodromy preserving deformation

In this section, following the recipe of [1, 2] we discuss the connection with the monodromy preserving deformation. From now on, we use the parameter $\hbar = 1/\kappa$ in place of $\kappa$.

Let $U$ be an invertible matrix solution to the confluent KZ equation (4.6), (4.7), and let $\tilde{U}(z)$ be the matrix solution to the extended system obtained by adjoining the point $z_0 = z$ and the two-dimensional $\mathfrak{g}$-module $\mathbb{C}^2$ with $r_0 = 0$. Then it is immediate to see that the quantity $Y(z) = U^{-1}\tilde{U}(z)$ satisfies the equations

$$\frac{\partial}{\partial z} Y = A(z)Y, \tag{5.1}$$

$$\frac{\partial}{\partial z_i} Y = B_{-1}^{(i)}(z)Y, \tag{5.2}$$

$$D_k^{(\lambda)} Y = B_k^{(\lambda)}(z)Y. \tag{5.3}$$

Here we have set

$$A(z) = \hbar U^{-1}L(z)U = \sum_{i=1}^{n}\sum_{p=0}^{r_i} \frac{A_p^{(i)}}{(z-z_i)^{p+1}} - \sum_{p=1}^{r_\infty} z^{p-1}A_p^{(\infty)},$$

$$B_k^{(i)}(z) = -\sum_{p=0}^{r_i-k-1} \frac{A_{p+k+1}^{(i)}}{(z-z_i)^{p+1}},$$

$$B_k^{(\infty)}(z) = \begin{cases} -\sum_{p=0}^{r_\infty-k} z^p A_{p+k}^{(\infty)} + \frac{1}{2}\sigma^3\gamma_k^{(\infty)} & (k \geqslant 1), \\ zA(z) + \sum_{i=1}^{n}\left(z_i B_{-1}^{(i)}(z) + B_0^{(i)}(z)\right) - \frac{\hbar}{4} - \frac{\hbar}{2}(h_0+1)\sigma^3 & (k = 0). \end{cases}$$

The coefficient matrices in (5.1)–(5.3) have the form

$$A_k^{(\lambda)} = \begin{pmatrix} \frac{1}{2}\bar{h}_k^{(\lambda)} & \bar{f}_k^{(\lambda)} \\ \bar{e}_k^{(\lambda)} & -\frac{1}{2}\bar{h}_k^{(\lambda)} \end{pmatrix},$$

where the entries satisfy the rescaled commutation relations

$$\left[\bar{\xi}_k^{(\lambda)}, \bar{\eta}_l^{(\mu)}\right] = \hbar\delta_{\lambda,\mu}\bar{\zeta}_{k+l}^{(\lambda)}, \qquad \text{with} \quad \zeta = [\xi, \eta].$$

The integrability condition for (5.1)–(5.3) gives rise to a system of nonlinear differential equations for them with respect to the 'time' variables $\mathbf{z}$ and $\gamma$. These are the quantization of the (irregular) Schlesinger system.

**Lemma 5.1.** *The quantized Schlesinger system given above are Hamiltonian equations. The Hamiltonians corresponding to the vector fields $\partial/\partial z_i$ and $D_k^{(\lambda)}$ are given by*

$$H_k^{(i)} = \hbar U^{-1}G_k^{(i)}U \qquad (1 \leqslant i \leqslant n, -1 \leqslant k \leqslant r_i - 1),$$

$$H_k^{(\infty)} = \begin{cases} \hbar U^{-1}G_k^{(\infty)}U + \frac{\hbar}{2}h_0\gamma_k^{(\infty)} & (1 \leqslant k \leqslant r_\infty - 1), \\ \hbar U^{-1}G_0^{(\infty)}U - \hbar\frac{h_0}{2}\left(\frac{h_0}{2}+1\right) & (k = 0). \end{cases}$$

**Proof.** The proof being similar, we consider the case of $D_k^{(i)}$ $(1 \leqslant i \leqslant n, k \geqslant 0)$.

In the commutation relation (2.2), the singular part at $w = z_i$ gives

$$\left[G_k^{(i)}, L(z)\right] = \left[\sum_{p\geqslant 0} \frac{\Omega_{0,p+k+1}^{(0,i)}}{(z-z_i)^{p+1}}, L(z)\right].$$

Conjugating with $U^{-1}$ we obtain

$$\left[H_k^{(i)}, A(z)\right] = -\left[B_k^{(i)}(z), A(z)\right].$$

Hence the integrability condition implies

$$0 = \left[\frac{\partial}{\partial z} - A(z), D_k^{(i)} - B_k^{(i)}(z)\right] = D_k^{(i)} A(z) - \frac{\partial B_k^{(i)}}{\partial z} - \left[A(z), H_k^{(i)}\right].$$

We now note that the second term can be written as

$$\frac{\partial B_k^{(i)}}{\partial z} = \sum_{p\geqslant 0}(p+1)\frac{A_{p+k+1}^{(i)}}{(z-z_i)^{p+2}} = \mathcal{D}_k^{(i)} A(z),$$

where $\mathcal{D}_k^{(i)}$ means $D_k^{(i)}$ acting only on the time variables, regarding the dynamical variables as constant. In summary we have obtained

$$D_k^{(i)} A(z) = \left[A(z), H_k^{(i)}\right] + \mathcal{D}_k^{(i)} A(z),$$

showing that $\left\{H_k^{(i)}\right\}_{0\leqslant k\leqslant r_i-1}$ are the Hamiltonians for the time flow with respect to $\gamma^{(i)}$. □

**Remark.** The symplectic structure of the isomonodromy deformation equations has been studied in [10]. Apparently the explicit formula for the corresponding Hamiltonians is not known in general. For the Lie algebra $\mathfrak{gl}_n$, formulae for the Gaudin Hamiltonians in [11, 12] may help in this regard.

## 6. Examples: quantum Painlevé equations

In the previous section, we considered the quantum Schlesinger system for $\bar{\xi}_p$, which are linear operators defined on each (finite-dimensional) graded component of $\mathbf{M}(\gamma)$. In this section, we write them allowing for formal inverse of $\bar{\xi}_p$'s. Namely, we consider the quotient skew field generated by the latter. Calculating examples as in [15], we reproduce the quantization of Painlevé equations with the affine Weyl group symmetries. These equations have been found previously by one of the present authors [9] as a quantization of the Noumi–Yamada system [16].

*The case of* $\mathrm{QP_V}$. Let $n = 2, r_1 = r_2 = 0, z_1 = 0, z_2 = 1$ and $r_\infty = 1$. We set

$$e_1^{(\infty)} = f_1^{(\infty)} = 0, \qquad h_1^{(\infty)} = -\kappa\eta, \qquad \gamma_1^{(\infty)} = -\kappa\eta.$$

It is easy to see that the quantities

$$\bar{h}_0^{(1)} + \bar{h}_0^{(2)},$$

$$C_i = \tfrac{1}{2}\bar{h}_0^{(i)}\bar{h}_0^{(i)} + \bar{e}_0^{(i)}\bar{f}_0^{(i)} + \bar{f}_0^{(i)}\bar{e}_0^{(i)} \qquad (i = 1, 2),$$

are the first integrals. We shall reduce the quantum Schlesinger equation in terms of the coordinates which are invariant under the gauge transformation $A_0^{(i)} \to XA_0^{(i)}X^{-1}$ by a diagonal matrix $X$. More specifically, setting

$$\bar{h}_0^{(1)} + \bar{h}_0^{(2)} = -\theta_\infty - 2\hbar, \qquad C_i = \tfrac{1}{2}(\theta_i - \hbar)(\theta_i + \hbar),$$

we parametrize the entries as follows:

$$\bar{h}_0^{(1)} = 2\hat{\mu} + \theta_0 - \hbar, \qquad \bar{h}_0^{(2)} = -2\hat{\mu} - \theta_0 - \theta_\infty - \hbar,$$
$$\bar{f}_0^{(1)} = -(\hat{\mu} + \theta_0)\hat{u}, \qquad \bar{f}_0^{(2)} = \left(\hat{\mu} + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right)\hat{u}\hat{\lambda},$$
$$\bar{e}_0^{(1)} = \hat{u}^{-1}\hat{\mu}, \qquad \bar{e}_0^{(2)} = -(\hat{u}\hat{\lambda})^{-1}\left(\hat{\mu} + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\right).$$

The commutation relations for the new variables $\hat{\lambda}, \hat{\mu}, \hat{u}, \theta_0, \theta_1$ and $\theta_\infty$ derived from those of $\bar{\xi}_0^{(i)}$ read

$$[\hat{u}, \hat{\lambda}] = 0, \qquad [\hat{u}, \hat{\mu}] = \hbar\hat{u}, \qquad [\theta_\infty, \hat{\lambda}] = [\theta_\infty, \hat{\mu}] = 0,$$
$$[\hat{\lambda}, \hat{\mu}] = -\hbar\hat{\lambda}, \qquad [\theta_\infty, \hat{u}] = 2\hbar\hat{u},$$

and $\theta_0, \theta_1$ are central. The corresponding quantized Schlesinger system is

$$\eta\frac{\partial}{\partial\eta}\hat{\lambda} = \eta\hat{\lambda} - 2(\hat{\lambda} - 1)\hat{\mu}(\hat{\lambda} - 1) - \left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}\hat{\lambda} - \frac{3\theta_0 + \theta_1 + \theta_\infty}{2}\right)(\hat{\lambda} - 1),$$
$$\eta\frac{\partial}{\partial\eta}\hat{\mu} = \left(\hat{\mu} + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right)\hat{\lambda}\hat{\mu} - (\hat{\mu} + \theta_0)\hat{\lambda}^{-1}\left(\hat{\mu} + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\right),$$
$$\eta\frac{\partial}{\partial\eta}\hat{u}\cdot\hat{u}^{-1} = \left(\hat{\mu} + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right)\hat{\lambda} + \hat{\lambda}^{-1}\left(\hat{\mu} + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\right) - (2\hat{\mu} + \theta_0),$$

and $\theta_0, \theta_1, \theta_\infty$ are constants, with the Hamiltonian $H_0^{(\infty)}$. In order to compare these with the quantum fifth Painlevé equation in [9], we make a further change of variables. For the parameters we set

$$\alpha_0 = 2C - \frac{\theta_0 - \theta_1 + \theta_\infty}{2}, \qquad \alpha_1 = -C - \beta + \theta_0, \qquad \alpha_2 = -C - \alpha_1 + \beta + \frac{\theta_0 + \theta_1 + \theta_\infty}{2},$$
$$\alpha_3 = 1 - \alpha_0 - \alpha_1 - \alpha_2, \qquad h = -\hbar,$$

where $C$ and $\beta$ are determined by

$$C^2 - \frac{\theta_0 - \theta_1 + \theta_\infty}{2}C + \frac{h(\theta_0 - \theta_1 + \theta_\infty - h)}{4} = 0,$$
$$\beta\left(\frac{(-\theta_0 + \theta_1 + \theta_\infty)}{2} + \beta - h\right) + \frac{h}{2}\left(\frac{(-\theta_0 + \theta_1 + \theta_\infty)}{2} + 2\beta - \frac{h}{2}\right) = 0.$$

As for the functions $\hat{\lambda}, \hat{\mu}$, let

$$\hat{f}_0 = \frac{\eta^{\frac{1}{2}}}{1 - \hat{\lambda}}, \qquad \hat{f}_1 = (1 - \hat{\lambda})\frac{\bar{\mu}\hat{\lambda}^{-1} + \hat{\lambda}^{-1}\bar{\mu}}{2\eta^{\frac{1}{2}}}(1 - \hat{\lambda}),$$
$$\hat{f}_2 = \eta^{\frac{1}{2}} - \hat{f}_0, \qquad \hat{f}_3 = \eta^{\frac{1}{2}} - \hat{f}_1,$$
$$\bar{\mu} = \hat{\mu} + C - \frac{\alpha_1}{\hat{\lambda} - 1}.$$

Writing the equations in these new variables, we have the quantum fifth Painlevé equation

$$[\hat{f}_i, \hat{f}_{i+1}] = h, \qquad [\hat{f}_i, \hat{f}_{i+2}] = 0,$$
$$\eta\frac{\partial}{\partial\eta}\hat{f}_i = \hat{f}_i\hat{f}_{i+1}\hat{f}_{i+2} - \hat{f}_{i+2}\hat{f}_{i+3}\hat{f}_i + \left(\frac{1}{2} - \alpha_i\right)\hat{f}_{i+2} + \alpha_i\hat{f}_{i+2} \qquad (i \in \mathbb{Z}/4\mathbb{Z}),$$

which has an action of the affine Weyl group of type $A_3^{(1)}$ as Bäcklund transformations.

The other Painlevé equations are derived in a similar manner. In what follows, we give the choices for $n, r_i, \gamma_i^{(j)}$, transformations of $\bar{\xi}_i^{(j)}$, and Hamiltonians.

*The case of* $\mathrm{QP_{IV}}$. Let $n = 1, r_1 = 0, z_1 = 0, r_\infty = 2$ and

$$\gamma_1^{(\infty)} = -2\kappa\eta, \qquad \gamma_2^{(\infty)} = -2\kappa,$$
$$h_1^{(\infty)} = -2\kappa\eta, \qquad e_2^{(\infty)} = 0, \qquad f_2^{(\infty)} = 0, \qquad h_2^{(\infty)} = -2\kappa.$$

We transform the variables as

$$\bar{h}_0^{(1)} = \left(\tfrac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) + 2\theta_0\right), \qquad \bar{e}_0^{(1)} = -\hat{u}^{-1}\hat{p}\left(\tfrac{1}{2}\hat{q}\hat{p} + 2\theta_0\right), \qquad \bar{f}_0^{(1)} = \tfrac{1}{2}\hat{u}\hat{q},$$
$$\bar{e}_1^{(\infty)} = \hat{u}^{-1}(\hat{p}\hat{q} + 2\theta_0 + 2\theta_\infty), \qquad \bar{f}_1^{(\infty)} = -\hat{u}.$$

The commutation relations derived from $\bar{\xi}_i^{(j)}$ read

$$[\theta_\infty, \hat{q}] = 0, \qquad [\theta_\infty, \hat{p}] = 0, \qquad [\theta_\infty, \hat{u}] = \hbar\hat{u},$$
$$[\hat{u}, \hat{q}] = 0, \qquad [\hat{u}, \hat{p}] = 0, \qquad [\hat{p}, \hat{q}] = -2\hbar,$$

and $\theta_0$ is central. The Hamiltonian is

$$\begin{aligned} H_1^{(\infty)} &= \hbar U^{-1} G_1^{(\infty)} U + \frac{\hbar}{2} h_0 \gamma_2^{\infty} = -\hbar U^{-1}\Omega_{0,1}^{(1,\infty)} U - \frac{1}{\hbar}(2\theta_\infty + \hbar) \\ &= -\frac{1}{2\hbar}\left(\hat{q}\hat{p}\hat{q} + \hat{p}\hat{q}\hat{p} - 2\eta\hat{p}\hat{q} + 2\theta_0(2\hat{p} + \hat{q}) + 2\theta_\infty\hat{q} - 2\eta(2\theta_0 - \hbar) + 2(2\theta_\infty + \hbar)\right) \\ &= \frac{1}{2\hbar}\left(\widehat{H}_{IV} - \eta\left(\frac{2\alpha_1}{3} + \frac{\alpha_2}{3} + 2\hbar\right) - 2\alpha_1 - \alpha_2 - 2\hbar\right), \end{aligned}$$

where $\widehat{H}_{IV}$ is the Hamiltonian of the quantum fourth Painlevé equation in [9] where

$$\hat{f}_0 = 2\eta - \hat{p} - \hat{q}, \qquad \hat{f}_1 = \hat{p}, \qquad \hat{f}_2 = \hat{q},$$
$$\alpha_0 = 2(1 + \theta_0 - \theta_\infty), \qquad \alpha_1 = 2(\theta_0 + \theta_\infty), \qquad \alpha_2 = -4\theta_0.$$

*The case of* $\mathrm{QP_{III}}$. Let us first present a quantization of $\mathrm{P_{III}}$ with affine Weyl group symmetry, as it has not been discussed in [9]. Let $\mathcal{K}_{\mathrm{III}}$ be a skew field with generators $\hat{p}, \hat{q}, \alpha_1, \alpha_2$ and $\eta$, and with commutation relations $[\hat{p}, \hat{q}] = h$ $(h \in \mathbb{C})$ and $\alpha_1, \alpha_2, t$ are central. We define a quantized Hamiltonian $\widehat{H}_{\mathrm{III}}$ by

$$\begin{aligned} \widehat{H}_{\mathrm{III}} = \tfrac{1}{4}&\left(\hat{p}\hat{q}(\hat{p} - 1)\hat{q} + (\hat{p} - 1)\hat{q}\hat{p}\hat{q} + \hat{q}\hat{p}\hat{q}(\hat{p} - 1) + \hat{q}(\hat{p} - 1)\hat{q}\hat{p}\right) \\ &+ \tfrac{1}{2}(\alpha_0 + \alpha_2)(\hat{q}\hat{p} + \hat{p}\hat{q}) - \alpha_0\hat{q} + \eta\hat{p}, \end{aligned}$$

where $\alpha_0 = 1 - 2\alpha_1 - \alpha_2$. We define a $\mathbb{C}$-derivation $\delta_\eta$ by

$$\delta_\eta(f) = \frac{1}{h}[\widehat{H}_{\mathrm{III}}, f] + \eta\frac{\partial f}{\partial \eta} \qquad (f \in \mathcal{K}_{\mathrm{III}}).$$

The Heisenberg equations for $\hat{p}$ and $\hat{q}$ are

$$\delta_\eta(\hat{q}) = \frac{1}{h}[\widehat{H}_{\mathrm{III}}, \hat{q}] = 2\hat{q}\hat{p}\hat{q} - \hat{q}^2 + (\alpha_0 + \alpha_2)\hat{q} + \eta,$$
$$\delta_\eta(\hat{p}) = \frac{1}{h}[\widehat{H}_{\mathrm{III}}, \hat{p}] = -2\hat{p}\hat{q}\hat{p} + \hat{q}\hat{p} + \hat{p}\hat{q} - (\alpha_0 + \alpha_2)\hat{p} + \alpha_0.$$

We define further the transformations for the generators of $\mathcal{K}_{\mathrm{III}}$ as follows:

| | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\eta$ | $\hat{q}$ | $\hat{p}$ |
|---|---|---|---|---|---|---|
| $s_0$ | $-\alpha_0$ | $\alpha_1 + \alpha_0$ | $\alpha_2$ | $\eta$ | $\hat{q} + \alpha_0\hat{p}^{-1}$ | $\hat{p}$ |
| $s_1$ | $\alpha_0 + 2\alpha_1$ | $-\alpha_1$ | $\alpha_2 + 2\alpha_1$ | $-\eta$ | $\hat{q}$ | $\hat{p} - 2\alpha_1\hat{q}^{-1} + \eta\hat{q}^{-2}$ |
| $s_2$ | $\alpha_0$ | $\alpha_1 + \alpha_2$ | $-\alpha_2$ | $\eta$ | $\hat{q} + \alpha_2(\hat{p} - 1)^{-1}$ | $\hat{p}$ . |

Then $s_i$'s preserve the commutation relations and satisfy the relations

$$(s_i)^2 = 1, \qquad (s_0 s_1)^4 = 1, \qquad (s_1 s_2)^4 = 1.$$

Hence $W = \langle s_0, s_1, s_2 \rangle$ gives a representation of the affine Weyl group of type $C_2^{(1)}$. Moreover, $s_i$ $(i = 0, 1, 2)$ commute with the derivation $\delta_\eta$.

Now, we turn to the confluent KZ equation. Let $n = 1, r_1 = 1, z_1 = 0, r_\infty = 1$ and

$$e_1^{(\infty)} = f_1^{(\infty)} = 0, \qquad h_1^{(\infty)} = -\kappa\eta, \qquad \gamma_1^{(\infty)} = -\kappa\eta, \qquad \gamma_1^{(1)} = -\kappa\eta.$$

We change variables as

$$\bar{e}_0^{(1)} = (wp)^{-1}\left(\frac{\theta_0}{2} + \frac{1}{2}\bar{h}_0^{(1)}\bar{h}_1^{(1)} + \bar{e}_1^{(1)}\bar{f}_0^{(1)}\right), \qquad \bar{f}_0^{(1)} = -\hat{q}w\hat{p}, \qquad \bar{h}_0^{(1)} = -\theta_\infty,$$

$$\bar{e}_1^{(1)} = w^{-1}(\hat{p} - 1), \qquad \bar{f}_1^{(1)} = -w\hat{p}, \qquad \bar{h}_1^{(1)} = 2\hat{p} - 1.$$

Their commutation relations derived from $\bar{\xi}_i^{(j)}$ are

$$[\theta_\infty, \hat{q}] = 0, \qquad [\theta_\infty, \hat{p}] = 0, \qquad [\theta_\infty, w] = 2\hbar w,$$
$$[w, \hat{q}] = \hbar w\hat{p}^{-1}, \qquad [w, \hat{p}] = 0, \qquad [\hat{p}, \hat{q}] = -\hbar,$$

and $\theta_0$ is central. The Hamiltonian is

$$H_0^{(1)} = \hbar U^{-1} G_0^{(1)} U = \hbar U^{-1}\left(\frac{1}{2}\Omega_{0,0}^{(1,1)} - \Omega_{1,1}^{(1,\infty)}\right)U$$
$$= \frac{1}{\hbar}\left(\hat{q}(\hat{p} - 1)\hat{q}\hat{p} - \hbar\hat{q}(1 - \hat{p}) + \theta_\infty\hat{q}\hat{p} + \frac{\theta_0 + \theta_\infty}{2}\hat{q} + \eta\hat{p} + \frac{\theta_\infty^2}{4} - \frac{\eta}{2} + \frac{\hbar\theta_\infty}{2}\right)$$
$$= \frac{1}{\hbar}\left(\widehat{H}_{\mathrm{III}} - \frac{\hbar^2}{2} + \frac{(\alpha_0 + \alpha_2)^2}{4} - \frac{\eta}{2}\right),$$

where $\alpha_0 = (\theta_0 + \theta_\infty)/2$ and $\alpha_2 = -(\theta_0 - \theta_\infty)/2$.

*The case of* $\mathrm{QP_{II}}$. Let $n = 0, r_\infty = 3$ and

$$\gamma_1^{(\infty)} = -\kappa\eta, \qquad \gamma_2^{(\infty)} = 0, \qquad \gamma_3^{(\infty)} = -2\kappa,$$
$$e_3^{(\infty)} = f_3^{(\infty)} = 0, \qquad h_3^{(\infty)} = -2\kappa, \qquad h_2^{(\infty)} = 0.$$

We change variables as

$$\bar{e}_1^{(\infty)} = 2\hat{u}^{-1}(2\hat{p}\hat{q} + \theta), \qquad \bar{f}_1^{(\infty)} = \hat{u}\hat{q}, \qquad \bar{h}_1^{(\infty)} = -4\hat{p} - \eta$$
$$\bar{e}_2^{(\infty)} = 4\hat{u}^{-1}\hat{p}, \qquad \bar{f}_2^{(\infty)} = -\hat{u}.$$

The commutation relations derived from $\bar{\xi}_i^{(j)}$ are

$$[\theta, \hat{q}] = 0, \qquad [\theta, \hat{p}] = 0, \qquad [\theta, \hat{u}] = \hbar\hat{u},$$
$$[\hat{u}, \hat{q}] = 0, \qquad [\hat{u}, \hat{p}] = 0, \qquad [\hat{p}, \hat{q}] = -\frac{\hbar}{2}.$$

The Hamiltonian is

$$H_2^{(\infty)} = \hbar U^{-1} G_2^{(\infty)} U = \frac{\hbar}{2} U^{-1}\Omega_{1,1}^{(\infty,\infty)} U = \frac{4}{\hbar}\left(\hat{q}\hat{p}\hat{q} + \hat{p}^2 + \frac{\eta}{2}\hat{p} + \frac{\theta}{2}\hat{q}\right) = -\frac{4}{\hbar}\widehat{H}_{\mathrm{II}},$$

where $\widehat{H}_{\mathrm{II}}$ is the Hamiltonian of the quantum second Painlevé equation in [9] where

$$\hat{f}_0 = -\frac{\eta}{2} - \hat{p} - \hat{q}^2, \qquad \hat{f}_1 = \hat{p}, \qquad \hat{f}_2 = \hat{q}, \qquad \alpha_1 = -\theta/2.$$

*The case of* $\mathrm{QP_I}$. This case corresponds to a representation of $t\mathfrak{g}[t]/t^4\mathfrak{g}[t]$ such that

$$e_3^{(\infty)} = 0, \qquad f_3^{(\infty)} = 1, \qquad h_3^{(\infty)} = 0.$$

Such modules do not belong to the confluent Verma modules $M'(\gamma')$ in section 3. In both cases, the central elements $\xi \otimes t^r$ of $\mathfrak{g}'_{(r)}$ act as a scalar $\chi(\xi)$, where $\chi$ is a linear form $t^r\mathfrak{g}[t]/t^{r+1}\mathfrak{g}[t] \to \mathbb{C}$. Under the identification $(t^r\mathfrak{g}[t]/t^{r+1}\mathfrak{g}[t])^* \simeq \mathfrak{g}^* \simeq \mathfrak{g}$, $\chi$ for $M'(\gamma')$ corresponds to a semi-simple element $\gamma_r h \in \mathfrak{g}$, while here it is a nilpotent element $e \in \mathfrak{g}$. For such modules the preceding considerations do not apply directly, and we do not know how to write the KZ equation in general and their integral solutions.

Here we are content to formally deriving $\mathrm{QP_I}$ by considering the system

$$\frac{\partial Y}{\partial z} = \left(\sigma^+ z^2 + A_2^{(\infty)} z + A_1^{(\infty)}\right) Y,$$

$$\frac{\partial Y}{\partial \eta} = \left(\sigma^+ z + A_2^{(\infty)}\right) Y,$$

where we set $\eta = \gamma_1^{(\infty)}$,

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_p^{(\infty)} = \begin{pmatrix} \bar{h}_p/2 & \bar{f}_p \\ \bar{e}_p & -\bar{h}_p/2 \end{pmatrix},$$

and $\bar{\xi}_p$'s satisfy the commutation relations of $t\mathfrak{g}[t]/t^4\mathfrak{g}[t]$ scaled by $\hbar$. With the change of variables

$$\bar{e}_1^{(\infty)} = \hat{q}, \qquad \bar{f}_1^{(\infty)} = -\tfrac{1}{2}(2\hat{u}\hat{p} + \hat{u}^2\hat{q} + 4\hat{q}^2) - \eta - \sqrt{2}C, \qquad \bar{h}_1^{(\infty)} = \sqrt{2}(\hat{p} + \hat{u}\hat{q}),$$

$$\bar{e}_2^{(\infty)} = -\sqrt{\tfrac{1}{2}}, \qquad \bar{f}_2^{(\infty)} = \sqrt{\tfrac{1}{2}}\left(\tfrac{1}{2}\hat{u}^2 - 2\hat{q}\right), \qquad \bar{h}_2^{(\infty)} = \hat{u},$$

the commutation relations become

$$[C, \hat{q}] = 0, \qquad [C, \hat{p}] = 0, \qquad [C, \hat{u}] = -\hbar\sqrt{2},$$

$$[\hat{u}, \hat{q}] = 0, \qquad [\hat{u}, \hat{p}] = 0, \qquad [\hat{p}, \hat{q}] = -\hbar.$$

The Hamiltonian is

$$H_2^{(\infty)} = \hbar U^{-1} G_2^{(\infty)} U = \frac{\hbar}{2} U^{-1} \Omega_{1,1}^{(\infty,\infty)} U = \frac{1}{\hbar}\left(\frac{\hat{p}^2}{2} - 2\hat{q}^3 - (\eta + \sqrt{2}C)\hat{q}\right).$$

Note that $C$ commutes with $\hat{p}, \hat{q}$ and $H_2^{(\infty)}$. Redefining $\eta$ we obtain a quantization of the Hamiltonian

$$H_I(q, p, t) = \tfrac{1}{2}p^2 - 2q^3 - tq$$

for the first Painlevé equation.

## Acknowledgments

Research of MJ is supported by the Grant-in-Aid for Scientific Research B-18340035. MJ and HN are grateful to Koji Hasegawa, Gen Kuroki and Dmitri Talalaev for helpful discussions.

## References

[1] Reshetikhin N 1992 The Knizhnik–Zamolodchikov system as a deformation of the isomonodromy problem *Lett. Math. Phys.* **26** 167–77

[2] Harnad J 1996 Quantum isomonodromic deformations and the Knizhnik–Zamolodchikov equations *Symmetries and Integrability of Difference Equations (Estérel, PQ, 1994) CRM Proc. Lecture Notes* vol 9*)* (Providence, RI: American Mathematical Society) pp 155–61

[3] Jimbo M, Miwa T and Ueno K 1981 Monodromy preserving deformation of linear ordinary differential equations with rational coefficients: I. General theory and $\tau$ function *Physica* D **2** 306–52

[4] Felder G, Markov Y, Tarasov V and Varchenko A 2000 Differential equations compatible with KZ equations *Math. Phys. Anal. Geom.* **3** 139–77
[5] Milsson J and Toledano Laredo V 2005 Casimir operators and monodromy representations of generalized braid groups *Transf. Groups* **10** 217–54
[6] Boalch P 2002 G-bundles, isomonodromy and quantum Weyl groups *Int. Math. Res. Not.* **22** 1129–66
[7] Feigin B, Frenkel E and Toledano Laredo V 2006 Gaudin models with irregular singularities *Preprint* math.QA/06127981
[8] Feigin B, Frenkel E and Rybnikov L 2007 Opers with irregular singularity and spectra of the shift of argument subalgebra *Preprint* 0712.1183
[9] Nagoya H 2004 Quantum Painlevé Systems of Type $A_l^{(1)}$ *Int. J. Math.* **15** 1007–31
[10] Boalch P 2001 Symplectic manifolds and isomonodromic deformations *Adv. Math.* **163** 137–205
[11] Talalaev D 2006 Quantization of the Gaudin system *Funct. Anal. Appl.* **40** 86–91
[12] Tarasov A 2000 On some commutative subalgebras in the universal enveloping algebra of the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ *Sb. Math.* **191** 1375–82
[13] Schechtman and Varchenko A 1989 Integral representations of *N*-point conformal correlators in the WZW model (Bonn: Max-Planck Insitut) pp 1–22
[14] Date E, Jimbo M, Matsuo A and Miwa T 1989 Hypergeometric-type integrals and the $\mathfrak{sl}_2$ Knizhnik–Zamolodchikov equation *Yang–Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory* (Singapore: World Scientific)
[15] Jimbo M and Miwa T 1981 Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II *Physica* D **2** 407–48
[16] Noumi M and Yamada Y 1998 Higher order Painlevé equations of type $A_l^{(1)}$ *Funk. Ekvac.* **41** 483–503